Volume of Solids of Known cross-sections and Volumes of Revolutions

Disk Volume

$$\text{Disk Volume} = \pi \int_a^b [R(x)]^2 \, dx \text{ or } \pi \int_c^d [R(y)]^2 \, dy$$

Cone Example

Washer Volume

$$\text{Washer Volume} = \pi \int_a^b [R(x)]^2 - [r(x)^2] \, dx \text{ or } \pi \int_c^d [R(y)]^2 - [r(y)^2] \, dy$$

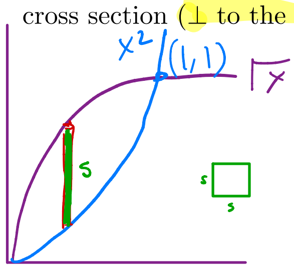
Volume of Solids with a Known Cross Section

$$\text{Volume of Solids with a Known Cross Sections} = \int_a^b A(x) \, dx \text{ or } \int_c^d A(y) \, dy$$

Where the most common cross sections are:

Square	Rectangle	Semicircle	Triangle	Equilateral $\triangle$	Iso.Rt. $\triangle$
$A = x^2$	$A = bh$	$A = \frac{1}{2}\pi r^2$	$A = \frac{1}{2}bh$	$A = \frac{\sqrt{3}}{4}s^2$	$A = \frac{(leg)^2}{2}$ leg in region $A = \frac{(hyp)^2}{4}$ hyp in region

1. To see an interactive 3-D rendering, go to `https://www.geogebra.org/m/XFgMaKTy`
   (a) Find the volume of the solid whose base is the enclosed area between $y = \sqrt{x}$ and $y = x^2$, whose cross section ($\perp$ to the x-axis) is a square ($A = s^2$)



$$\int_0^1 A(x)\, dx$$

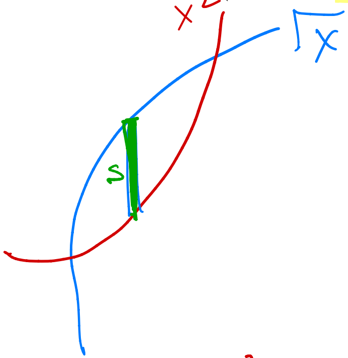
$$\int_0^1 (\sqrt{x} - x^2)^2 \, dx = \frac{9}{70} \quad (\approx 0.12857)$$

$A = s^2$

$s = (\sqrt{x} - x^2)$

$A = (\sqrt{x} - x^2)^2$

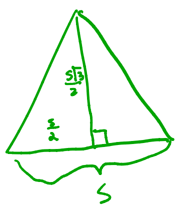
(b) Find the volume of the solid whose base is the enclosed area between $\boxed{y = \sqrt{x}}$ and $\boxed{y = x^2}$, whose cross section ($\perp$ to the x-axis) is a $\boxed{\text{Equilateral Triangle}}$ $\left(A = \frac{\sqrt{3}}{4}s^2\right)$



$$\frac{\sqrt{3}}{4}\int_0^1 \left(\sqrt{x} - x^2\right)^2 dx = \frac{9\sqrt{3}}{280}$$

$(\approx .055673)$

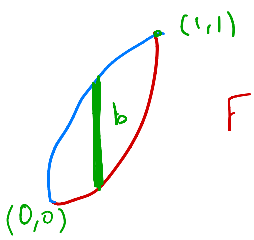
$s = \sqrt{x} - x^2$



$A = \frac{s^2\sqrt{3}}{4}$

$A = \frac{\sqrt{3}}{4} s^2$

(c) Find the volume of the solid whose base is the enclosed area between $\boxed{y = \sqrt{x}}$ and $\boxed{y = x^2}$, whose cross section ($\perp$ to the x-axis) is a $\boxed{\text{Semi-circle}}$ $\left(A = \frac{1}{2}\pi r^2\right)$



$r = \frac{d}{2}$

$$\frac{\pi}{8}\int_0^1 \left(\sqrt{x} - x^2\right)^2 dx =$$

$b = \sqrt{x} - x^2$

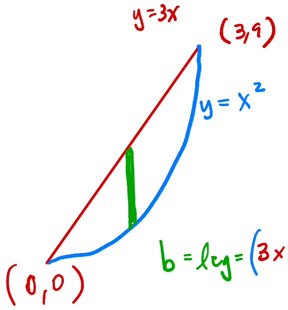
$$= \frac{\pi}{8}\left(\frac{9}{70}\right) = \frac{9\pi}{560}$$

$\left.\begin{array}{l} A = \frac{\pi r^2}{2} \\ \text{where} \\ r = \frac{d}{2} \end{array}\right\}$ so... $A = \frac{\pi}{2}\left(\frac{b}{2}\right)^2$

$A = \frac{\pi}{8} b^2$

2. Class 3-18 Examples from DeltaMath

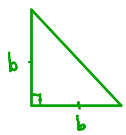
(a) Let the region R be the area enclosed by the function $f(x) = x^2$ and $g(x) = 3x$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is an **isosceles right triangle** with a leg in the region R, find the volume of the solid. You may use a calculator and round to the nearest thousandth.



$b = \text{leg} = (3x - x^2)$

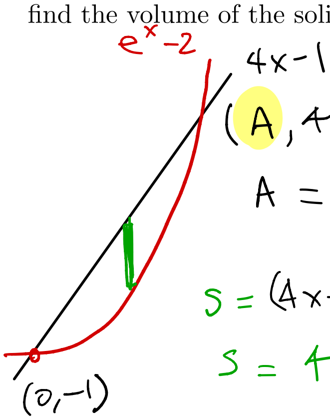
$$V = \int_0^3 A(x)\,dx = \int_0^3 \frac{b^2}{2}\,dx$$

$$V = \frac{1}{2}\int_0^3 (3x - x^2)^2\,dx = \frac{81}{20}$$



$A = \frac{b^2}{2}$

(b) Let the region R be the area enclosed by the function $f(x) = e^x - 2$ and $g(x) = 4x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is a **square**, find the volume of the solid. You may use a calculator and round to the nearest thousandth.



$(A, 4A-1)$

$A = 2.336662982$

$$V = \int_0^A (4x - e^x + 1)^2\,dx$$

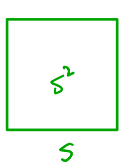
$s = (4x-1) - (e^x - 2)$

$s = 4x - e^x + 1$

$V = 7.913594707$

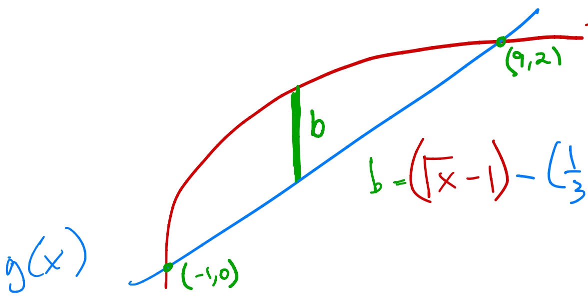
(7.914)

$A = s$



(c) Let the region R be the area enclosed by the function $f(x) = \sqrt{x} - 1$ and $g(x) = \frac{1}{3}x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is a **rectangle** whose height is half the length of its base in the region R, find the volume of the solid. You may use a calculator and round to the nearest thousandth.

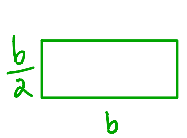
$f(x) = \sqrt{x} - 1$



$b = (\sqrt{x} - 1) - (\frac{1}{3}x - 1)$

$$V = \frac{1}{2}\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)^2 dx$$

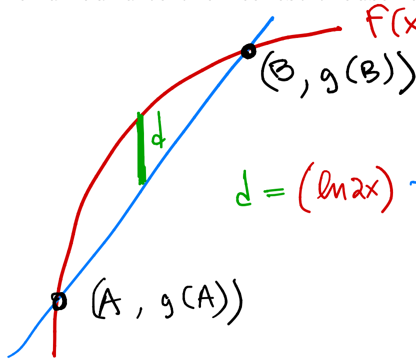
$$= \frac{27}{20} \text{ or } 1.35$$



$A = \frac{b^2}{2}$

(d) Let the region R be the area enclosed by the function $f(x) = \ln(2x)$ and $g(x) = \frac{1}{2}x - 1$. If the region R is the base of a solid such that each cross section perpendicular to the x-axis is a **semi-circle** with diameters extending through the region R, find the volume of the solid. You may use a calculator and round to the nearest thousandth.

$f(x) = \ln(2x)$



$d = (\ln 2x) - (\frac{x}{2} - 1)$

$g(x) = \frac{x}{2} - 1$

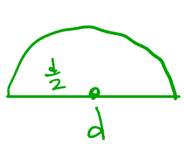
$$\frac{\pi}{8}\int_A^B \left(\ln(2x) - \frac{x}{2} + 1\right)^2 dx$$

where $A = 0.2036568622$

and $B = 7.385269058$

$$= 2.770248364$$

(2.770)



$A = \frac{\pi}{2}\left(\frac{d}{2}\right)^2 = \frac{\pi}{8} d^2$